Lecture 17 (Connectivity in graphs)

Separating set: Let $G = (V, E)$ be a graph. A set $V' \subseteq V$ is called a separating set if $G \setminus V'$ is disconnected. Similarly, a set $E' \subseteq E$ is called an edge cut of G if the subgraph $G - E'$ is disconnected.

If $\{u\} \subseteq V(G)$ is a separating set of G, then u is called a cut-vertex. If $\{e\} \subseteq E(G)$ is an edge cut of G, then it is called a bridge/cut-edge.

1. In a tree, each edge is a bridge and each non-pendant vertex is a cut-vertex.
2. The graph K_7 does not have a separating set of vertices. In K_7, a separating set of edges must contain at least 6 edges.

Vertex connectivity: The vertex connectivity of a non-complete graph G, denoted by $\kappa(G)$, is the minimum number of vertices in a separating set of G. A graph G is said to be k-vertex connected if $\kappa(G) \geq k$, i.e., G is connected even after deletion of any $k-1$ vertices

1. For a disconnected graph G, $\kappa(G) = 0$ and for $n > 1$, $\kappa(K_n) = n - 1$.
2. The Peterson graph is 3-connected.

Edge connectivity: The edge connectivity of a graph G, denoted by $\lambda(G)$, is the minimum number of edges in an edge cut of G. A graph G is called l-edge connected if $\lambda(G) \geq l$, i.e., for every $F \subseteq E(G)$, $|F| < l$, $G \setminus F$ is connected.

1. For $n > 1$, $\lambda(P_n) = 1$, $\lambda(C_n) = 2$, and $\lambda(K_n) = n - 1$.
2. A tree T with $n \geq 2$ vertices has $\lambda(T) = 1$.
3. For the Peterson graph $\lambda = 3$.

Theorem 0.1 For any graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

Proof: If G is disconnected graph or $|G| = 1$, then nothing to prove, as then $\kappa(G) = \lambda(G) = 0$. So let G is connected graph with $|G| \geq 2$. Then there is a vertex v whose $\deg(v) = \delta(G)$. Now if we remove all the edges incident at v, then G becomes disconnected. So $\delta(G) \geq \lambda(G)$.

Now we have two cases:

Suppose $\lambda(G) = 1$. Then there is an edge uv such that $G \setminus uv$ is disconnected and we get two components say C_u and C_v. If $|C_u| = |C_v| = 1$, then $G = K_2$ which gives $\kappa(G) = 1$. If any of them, say $|G_u| > 1$, then removing u, we get disconnected graph. Then $\kappa(G) = 1$.

Suppose $\lambda(G) = k \geq 2$. Clearly removal of $k-1$ of these lines produces a graph with a bridge $x = uv$. For each of these $k-1$ lines, select an incident point different from u or v. The removal of these points will also remove $k-1$ lines and quite possible more. If the resulting graph is disconnected, the $\kappa < \lambda$; if not, x is a bridge, and hence removal of u or v will result in either a disconnected or a trivial graph. So $\kappa \leq \lambda$ in every case.

Remark: One can demonstrate the above proof by considering the graph $G = (V, E)$, where $V = \{1, 2, 3, 4, 5, 6, 7\}$ and $E = \{12, 13, 23, 24, 25, 45, 56, 57, 67\}$. Consider $x = 25$ and $k-1$ vertex as 4.

Theorem 0.2 *For all integer a, b, c such that $0 < a \leq b \leq c$, there exists a graph G with $\kappa(G) = a$, $\lambda(G) = b$ and $\delta(G) = c$.*